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Positive solutions for singular boundary value problems of a coupled system of differential equations ☆

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Abstract

By using fixed point theorem of cone expansion and compression, this paper investigates the existence of multiple positive solutions for singular boundary value problems of a coupled system of nonlinear ordinary differential equations.

Keywords: Singular boundary value problem; Positive solution; Cone

1. Introduction

This paper consider the existence of multiple positive solutions for coupled singular system of second and fourth order ordinary differential equations

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$$\begin{cases} u^{(4)} = f(t, v), & (t, v) \in (0, 1) \times R^+, \\ -v'' = g(t, u), & (t, u) \in (0, 1) \times R^+, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \\ v(0) = v(1) = 0, \end{cases} \tag{1.1}$$

where $f \in C[(0,1) \times R^+, R^+]$, $g \in C[(0,1) \times R^+, R^+]$, that is, f, g are probably singular at $t = 0$ and $t = 1$.

In recent years, singular boundary value problems (BVP for short) to second or fourth order ordinary differential equations have been studied extensively (see, for example, [1–6] and references therein). Naturally we hope there are same excellent results on singular boundary value problems to the coupled system of second and fourth order ordinary differential equations. But as far as we know, we find only the following results on differential systems without singularity.

Recently, Agarwal and O'Regan [7,8] have considered differential systems. For example, in [8], using Leray–Schauder theory, they studied

$$\begin{cases} u''(t) + f(t, v(t)) = 0, & \text{a.e. } t \in [0, 1], \\ v''(t) + g(t, u(t)) = 0, & \text{a.e. } t \in [0, 1], \\ \alpha_1 u(0) - \beta_1 u'(0) = 0, \\ \gamma_1 u(1) + \delta_1 u'(1) = 0, \\ \alpha_2 v(0) - \beta_2 v'(0) = 0, \\ \gamma_2 v(1) + \delta_2 v'(1) = 0, \end{cases} \tag{1.2}$$

and obtained the existence of solutions of system (1.2).

Further, in a recent paper [9], Agarwal et al. investigated far reaching extensions for the above systems, that is, they consider the following system of Fredholm integral equations:

$$u_i(t) = \int_0^1 g_i(t, s) f_i\bigl(s, u_1(s), u_2(s), \ldots, u_n(s)\bigr)\, ds, \quad t \in [0, 1],\ 1 \leqslant i \leqslant n. \tag{1.3}$$

Criteria are offered for the existence of single, double and multiple solutions of the system (1.3) that are of constant signs. The generality of the results obtained is illustrated through applications to several well-known boundary value problems.

In [10,11], the authors used the Leray–Schauder fixed point theorem and critical point theory to obtain the existence of one solution of the system similar to (1.1).

When singularity occurs, Y. Liu and B. Yan [12] considered the existence of multiple solutions for system of two second order ordinary differential equations.

The problem (1.1) can be seen as steady state from the problem of suspension bridge equations

$$\begin{cases} y_{tt} + y_{xxxx} + \delta_1 y_t + k(y - z)^+ = w(x), & (t, x) \in (0, L) \times R, \\ z_{tt} - z_{xx} + \delta_2 z_t - k(y - z)^+ = h(x, t), & (t, x) \in (0, L) \times R, \\ y(0, t) = y(L, t) = y_{xx}(0, t) = y_{xx}(L, t) = 0, & t \in R, \\ z(0, t) = z(L, t) = 0, & t \in R, \end{cases} \tag{1.4}$$

which is presented in [13]. System (1.4) takes account of the fact that the coupling provided by the ties connecting the suspension cable to the deck of the road bed is fundamentally nonlinear. The variable z measures the displacement from equilibrium of the cable and the variable y measures the displacement of the road bed. The constant k is a spring constant of the ties.

Obviously, what we consider is more different from those in [7–12]. Our main features are as follows. First, system (1.1) possesses singularity, that is, f and g may be singular at $t = 0$ and $t = 1$. Second, the main tool used here is the fixed point theorem of cone expansion and compression and the results obtained is the existence of multiple solutions of system (1.1). Third, the system here consists of second and fourth order ordinary differential equations. The organization of this paper is as follows. We introduce some preliminaries in the rest of this section. In Section 2, our main results are stated and proved. Finally some examples are worked out to demonstrate our main results in Section 3.

$(u, v) \in C^4[(0, 1), R^+] \times C^2[(0, 1), R^+]$ is said to be a positive solution of BVP (1.1) if and only if (u, v) satisfies BVP (1.1) and $u(t) > 0$, $v(t) > 0$ for $t \in (0, 1)$.

Lemma 1 [14]. *Let K be a cone of real Banach space E, Ω_1, Ω_2 be bounded open sets of E, $\theta \in \bar{\Omega}_1 \subset \Omega_2$. Suppose that $A : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ is completely continuous such that one of the following two conditions is satisfied*:

(i) $\|Ax\| \leqslant \|x\|$ *for* $x \in K \cap \partial\Omega_1$; $\|Ax\| \geqslant \|x\|$ *for* $x \in K \cap \partial\Omega_2$.
(ii) $\|Ax\| \leqslant \|x\|$ *for* $x \in K \cap \partial\Omega_2$; $\|Ax\| \geqslant \|x\|$ *for* $x \in K \cap \partial\Omega_1$.

Then, A has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

2. Main results

For convenience, let us list the following assumptions.

(H_1) $f \in C[(0, 1) \times R^+, R^+]$, $g \in C[(0, 1) \times R^+, R^+]$, and

$$f(t, v) \leqslant p_1(t)q_1(v), \quad g(t, u) \leqslant p_2(t)q_2(u), \quad t \in (0, 1),\ u, v \in R^+,$$

where $p_i \in C[(0, 1), R^+]$, $q_i \in C[R^+, R^+]$, $i = 1, 2$, $q_2(0) = 0$, and

$$a =: \frac{1}{12}\int_0^1 (\tau - 2\tau^3 + \tau^4)p_1(\tau)\,d\tau < +\infty,$$

$$b =: \int_0^1 s(1 - s)p_2(s)\,ds < +\infty.$$

(H_2) There exist $r_1 \in (0, +\infty)$ and $r_2 \in (0, +\infty)$ with $r_1 r_2 \geqslant 1$ satisfying

$$\overline{\lim_{u\to 0^+}} \frac{q_1(u)}{u^{r_1}} < +\infty, \qquad \overline{\lim_{u\to 0^+}} \frac{q_2(u)}{u^{r_2}} = 0.$$

(H_3) There exist $l_1 \in (0, +\infty)$ and $l_2 \in (0, +\infty)$ with $l_1 l_2 \geqslant 1$ satisfying

$$\lim_{u \to +\infty} \min_{t \in [1/4, 3/4]} \frac{f(t,u)}{u^{l_1}} > 0, \qquad \lim_{u \to +\infty} \min_{t \in [1/4, 3/4]} \frac{g(t,u)}{u^{l_2}} = +\infty.$$

(H_4) There exist $\alpha_1 \in (0, +\infty)$ and $\alpha_2 \in (0, +\infty)$ with $\alpha_1 \alpha_2 \leqslant 1$ satisfying

$$\varlimsup_{u \to +\infty} \frac{q_1(u)}{u^{\alpha_1}} < +\infty, \qquad \varlimsup_{u \to +\infty} \frac{q_2(u)}{u^{\alpha_2}} = 0.$$

(H_5) There exist $\beta_1 \in (0, +\infty)$ and $\beta_2 \in (0, +\infty)$ with $\beta_1 \beta_2 \leqslant 1$ satisfying

$$\lim_{u \to 0^+} \min_{t \in [1/4, 3/4]} \frac{f(t,u)}{u^{\beta_1}} > 0, \qquad \lim_{u \to 0^+} \min_{t \in [1/4, 3/4]} \frac{g(t,u)}{u^{\beta_2}} = +\infty.$$

(H_6) There exists $N > 0$ such that

$$\sup_{u \in [0, Mb]} q_1(u) \leqslant \frac{N}{a},$$

where $M = \sup_{u \in [0,N]} q_2(u)$, a, b are the same as in (H_1).

Our main results are as follows.

Theorem 1. *Assume that* (H_1)–(H_3) *are satisfied. Then BVP* (1.1) *has at least one positive solution.*

Theorem 2. *Assume that* (H_1), (H_4), *and* (H_5) *hold. Then BVP* (1.1) *has at least one positive solution.*

Theorem 3. *Assume that* (H_1), (H_3), (H_5), *and* (H_6) *hold. Then BVP* (1.1) *has at least two positive solutions.*

Before proving the theorems mentioned above, we first list some preliminaries and prove a lemma. Obviously, $(u, v) \in C^4(0,1) \times C^2(0,1)$ is a solution of BVP (1.1) if and only if $(u, v) \in C[0,1] \times C[0,1]$ is a solution of the following nonlinear integral system:

$$\begin{cases} u(t) = \int_0^1 G(t,s) \int_0^1 G(s,\tau) f(\tau, v(\tau))\, d\tau\, ds, & t \in [0,1], \\ v(t) = \int_0^1 G(t,s) g(s, u(s))\, ds, & t \in [0,1], \end{cases} \tag{2.1}$$

where

$$G(t,s) = \begin{cases} t(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \\ s(1-t), & 0 \leqslant s \leqslant t \leqslant 1. \end{cases}$$

Evidently, the nonlinear integral system (2.1) can be rewritten as the following nonlinear integral equation:

$$u(t) = \int_0^1 G(t,s) \int_0^1 G(s,\tau) f\left(\tau, \int_0^1 G(\tau,\xi) g\big(\xi, u(\xi)\big)\, d\xi\right) d\tau\, ds, \quad \forall t \in [0,1].$$

Let $E = C[0,1]$, $\|u\| = \max_{0\leqslant t\leqslant 1}|u(t)|$, $P = \{u \in E\colon\ u(t) \geqslant 0,\ \forall t \in [0,1]\}$, $B_R = \{u \in E\colon\ \|u\| \leqslant R\}$ $(R > 0)$. Then $(E, \|\cdot\|)$ is a Banach space, and P is a cone in E. Define an operator A on P by

$$(Au)(t) = \int_0^1 G(t,s)\int_0^1 G(s,\tau) f\left(\tau, \int_0^1 G(\tau,\xi) g\big(\xi, u(\xi)\big)\,d\xi\right) d\tau\,ds,\quad \forall t \in [0,1]. \tag{2.2}$$

Lemma 2. *Assume that* (H_1) *holds. Then* $A : P \to P$ *is completely continuous.*

Proof. First, it is easy to see A is an operator from P into P. Next, we prove that A maps bounded set into bounded set.

Suppose $D \subset P$ is an arbitrary bounded subset, that is to say, there exists $M_1 > 0$ such that $\|u\| \leqslant M_1$ for $\forall u \in D$. Since for all $u \in D$, $\tau \in [0,1]$, we have

$$\begin{aligned}\int_0^1 G(\tau,\xi) g\big(\xi, u(\xi)\big)\,d\xi &\leqslant \int_0^1 G(\tau,\xi) p_2(\xi) q_2\big(u(\xi)\big)\,d\xi \\ &\leqslant M_2 \int_0^1 G(\xi,\xi) p_2(\xi)\,d\xi = M_2 b,\end{aligned} \tag{2.3}$$

where $M_2 = \sup_{u\in[0,M_1]} q_2(u)$, b is the same as in (H_1).

Thus, by (2.3), we get

$$\begin{aligned}\big|(Au)(t)\big| &= \left|\int_0^1 G(t,s)\int_0^1 G(s,\tau) f\left(\tau, \int_0^1 G(\tau,\xi) g\big(\xi, u(\xi)\big)\,d\xi\right) d\tau\,ds\right| \\ &\leqslant \int_0^1 G(t,s)\int_0^1 G(s,\tau) p_1(\tau) q_1\left(\int_0^1 G(\tau,\xi) g\big(\xi, u(\xi)\big)\,d\xi\right) d\tau\,ds \\ &\leqslant M_0 \int_0^1 G(t,s)\int_0^1 G(s,\tau) p_1(\tau)\,d\tau\,ds \\ &\leqslant M_0 \int_0^1\int_0^1 s(1-s) G(s,\tau) p_1(\tau)\,d\tau\,ds \\ &= \frac{M_0}{12}\int_0^1 (\tau - 2\tau^3 + \tau^4) p_1(\tau)\,d\tau \\ &= M_0 a < +\infty,\quad \forall u \in D,\ t \in [0,1],\end{aligned} \tag{2.4}$$

where $M_0 = \sup_{[0,M_2 b]} q_1(u)$, a is the same as in (H_1).

Therefore A maps the bounded set D into a bounded set. In the following, we prove that $A(D)$ is equicontinuous.

For any $t \in (0,1)$, $u \in D$, we have

$$\begin{aligned}
\left|(Au)'(t)\right| &= \left| -\int_0^t s \int_0^1 G(s,\tau) f\left(\tau, \int_0^1 G(\tau,\xi) g\bigl(\xi, u(\xi)\bigr)\, d\xi\right) d\tau\, ds \right.\\
&\quad \left. + \int_t^1 (1-s) \int_0^1 G(s,\tau) f\left(\tau, \int_0^1 G(\tau,\xi) g\bigl(\xi, u(\xi)\bigr)\, d\xi\right) d\tau\, ds \right|\\
&\leqslant \int_0^t s \int_0^1 G(s,\tau) p_1(\tau) q_1\left(\int_0^1 G(\tau,\xi) g\bigl(\xi, u(\xi)\bigr)\, d\xi\right) d\tau\, ds\\
&\quad + \int_t^1 (1-s) \int_0^1 G(s,\tau) p_1(\tau) q_1\left(\int_0^1 G(\tau,\xi) g\bigl(\xi, u(\xi)\bigr)\, d\xi\right) d\tau\, ds\\
&\leqslant M_0\left(\int_0^t \int_0^1 sG(s,\tau) p_1(\tau)\, d\tau\, ds + \int_t^1 \int_0^1 (1-s) G(s,\tau) p_1(\tau)\, d\tau\, ds\right)\\
&= M_0 h(t),
\end{aligned} \tag{2.5}$$

where

$$h(t) =: \int_0^t \int_0^1 sG(s,\tau) p_1(\tau)\, d\tau\, ds + \int_t^1 \int_0^1 (1-s) G(s,\tau) p_1(\tau)\, d\tau\, ds, \quad t \in (0,1).$$

Notice that

$$\begin{aligned}
\int_0^1 h(t)\, dt &= \int_0^1 \int_0^t \int_0^1 sG(s,\tau) p_1(\tau)\, d\tau\, ds\, dt + \int_0^1 \int_t^1 \int_0^1 (1-s) G(s,\tau) p_1(\tau)\, d\tau\, ds\, dt\\
&= \int_0^1 \int_s^1 \int_0^1 sG(s,\tau) p_1(\tau)\, d\tau\, dt\, ds + \int_0^1 \int_0^s \int_0^1 (1-s) G(s,\tau) p_1(\tau)\, d\tau\, dt\, ds\\
&= 2\int_0^1 \int_0^1 s(1-s) G(s,\tau) p_1(\tau)\, d\tau\, ds\\
&= \frac{1}{6}\int_0^1 (\tau - 2\tau^3 + \tau^4) p_1(\tau)\, d\tau < +\infty.
\end{aligned} \tag{2.6}$$

Then by (H$_1$), we have $h(t) \in L^1(0,1)$.Thus for any given $0 \leqslant t_1 \leqslant t_2 \leqslant 1$, and $\forall u \in D$, by (2.5), we obtain

$$\left|(Au)(t_2) - (Au)(t_1)\right| = \left|\int_{t_1}^{t_2} (Au)'(t)\,dt\right| \leqslant M_0 \int_{t_1}^{t_2} h(t)\,dt. \tag{2.7}$$

By virtue of (2.6), (2.7), and the absolute continuity of integral function, it follows that $A(D)$ is equicontinuous. This together with (2.4) and Ascoli–Arzela theorem guarantees $A(D)$ is relatively compact.

Now, we prove that A is continuous. Suppose $u_n, u \in P$ and $\|u_n - u\| \to 0$ $(n \to +\infty)$. Then there exists $M_3 > 0$ such that $\|u_n\| \leqslant M_3$ and $\|u\| \leqslant M_3$. From the above proof we know that $\{Au_n\}$ is relatively compact. In the following we prove $\|Au_n - Au\| \to 0$ $(n \to +\infty)$. In fact, if this is not true, then there exist ε_0 and $\{u_{n_k}\} \subset \{u_n\}$ such that $\|Au_{n_k} - Au\| \geqslant \varepsilon_0$ $(k = 1, 2, \ldots)$. Since $\{Au_n\}$ is relatively compact, there exists a subsequence of $\{Au_{n_k}\}$ which converges in P to some $y \in P$. No loss of generality, we may assume that $\{Au_{n_k}\}$ itself converges to y, that is, $\lim_{k\to+\infty} \|Au_{n_k} - y\| = 0$. Obviously, $(Au_{n_k})(t) \to y(t)$, $k \to +\infty$, $t \in [0,1]$.

Notice that

$$\begin{aligned} G(\tau,\xi) g\big(\xi, u_{n_k}(\xi)\big) &\leqslant G(\tau,\xi) p_2(\xi) q_2\big(u_{n_k}(\xi)\big) \\ &\leqslant M_4 \xi(1-\xi) p_2(\xi), \quad \forall \tau \in [0,1], \end{aligned} \tag{2.8}$$

where $M_4 = \sup_{u \in [0, M_3]} q_2(u)$. Thus,

$$\begin{aligned} &G(t,s)G(s,\tau) f\left(\tau, \int_0^1 G(\tau,\xi) g\big(\xi, u_{n_k}(\xi)\big)\,d\xi\right) \\ &\quad \leqslant G(t,s)G(s,\tau) p_1(\tau) q_1\left(\int_0^1 G(\tau,\xi) g\big(\xi, u_{n_k}(\xi)\big)\,d\xi\right) \\ &\quad \leqslant M_5 s(1-s) G(s,\tau) p_1(\tau), \quad \forall t \in [0,1], \end{aligned} \tag{2.9}$$

where $M_5 = \sup_{u \in [0, M_4 b]} q_1(u)$. This together with (H$_1$), (2.8), (2.9), and Lebesgue dominated theorem yields that

$$y(t) = \lim_{k \to +\infty} (Au_{n_k})(t) = (Au)(t), \quad t \in [0,1],$$

that is, $y = Au$. This is a contradiction.

Consequently, A is continuous.

To sum up, the conclusion of Lemma 2 follows. □

Proof of Theorem 1. Let

$$K = \left\{ u \in P \colon \min_{t \in [1/4, 3/4]} u(t) \geqslant \frac{1}{4}\|u\| \right\}.$$

It is easy to see K is a cone of E. Now we show $A(K) \subset K$.

For $\forall u \in K$, by virtue of (2.2), one can see

$$(Au)(t) \leqslant \int_0^1 \int_0^1 s(1-s)G(s,\tau) f\left(\tau, \int_0^1 G(\tau,\xi) g\big(\xi, u(\xi)\big)\, d\xi\right) d\tau\, ds, \quad t \in [0,1].$$

Therefore,

$$\|Au\| \leqslant \int_0^1 \int_0^1 s(1-s)G(s,\tau) f\left(\tau, \int_0^1 G(\tau,\xi) g\big(\xi, u(\xi)\big)\, d\xi\right) d\tau\, ds.$$

From the definition of $G(t,s)$, it is easy to see

$$G(t,s) \geqslant \frac{1}{4} G(s,s), \quad (t,s) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times [0,1].$$

Thus,

$$\begin{aligned}
&\min_{t\in[1/4,3/4]} (Au)(t) \\
&\quad = \min_{t\in[1/4,3/4]} \int_0^1 G(t,s) \int_0^1 G(s,\tau) f\left(\tau, \int_0^1 G(\tau,\xi) g\big(\xi, u(\xi)\big)\, d\xi\right) d\tau\, ds \\
&\quad \geqslant \frac{1}{4} \int_0^1 G(s,s) \int_0^1 G(s,\tau) f\left(\tau, \int_0^1 G(\tau,\xi) g\big(\xi, u(\xi)\big)\, d\xi\right) d\tau\, ds \\
&\quad \geqslant \frac{1}{4}\|Au\|, \quad \forall u \in K.
\end{aligned}$$

Hence,

$$A(K) \subset K. \tag{2.10}$$

By condition (H_2), we know that there exist $c_1 > 0$, $\varepsilon_1 \in (0,1)$, and $\delta \in (0,1)$ such that

$$q_1(u) \leqslant c_1 u^{r_1}, \quad u \in [0,1],$$
$$q_2(u) \leqslant \varepsilon_1 u^{r_2}, \quad u \in [0,\delta],$$

and satisfying

$$\varepsilon_1 \int_0^1 \xi(1-\xi) p_2(\xi)\, d\xi \leqslant 1,$$

$$c_1 \varepsilon_1^{r_1} \int_0^1 \int_0^1 s(1-s) G(s,\tau) p_1(\tau)\, d\tau\, ds \cdot \left(\int_0^1 \xi(1-\xi) p_2(\xi)\, d\xi\right)^{r_1} \leqslant 1.$$

Notice that

$$\begin{aligned}
&\int_0^1 G(\tau,\xi)g\big(\xi,u(\xi)\big)\,d\xi \\
&\quad\leqslant \int_0^1 G(\tau,\xi)p_2(\xi)q_2\big(u(\xi)\big)\,d\xi \leqslant \varepsilon_1 \int_0^1 G(\tau,\xi)p_2(\xi)u^{r_2}(\xi)\,d\xi \\
&\quad\leqslant \varepsilon_1 \int_0^1 \xi(1-\xi)p_2(\xi)\,d\xi \cdot \|u\|^{r_2} \leqslant \|u\|^{r_2} \leqslant 1, \quad \forall u \in \partial B_\delta \cap K,\ \tau \in [0,1].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
(Au)(t) &\leqslant \int_0^1 G(t,s) \int_0^1 G(s,\tau)p_1(\tau)q_1\left(\int_0^1 G(\tau,\xi)g\big(\xi,u(\xi)\big)\,d\xi\right) d\tau\,ds \\
&\leqslant c_1 \int_0^1 G(t,s) \int_0^1 G(s,\tau)p_1(\tau)\left(\int_0^1 G(\tau,\xi)g\big(\xi,u(\xi)\big)\,d\xi\right)^{r_1} d\tau\,ds \\
&\leqslant c_1 \int_0^1 G(t,s) \int_0^1 G(s,\tau)p_1(\tau)\left(\int_0^1 G(\tau,\xi)p_2(\xi)q_2\big(u(\xi)\big)\,d\xi\right)^{r_1} d\tau\,ds \\
&\leqslant c_1\varepsilon_1^{r_1} \int_0^1 G(t,s) \int_0^1 G(s,\tau)p_1(\tau)\left(\int_0^1 G(\tau,\xi)p_2(\xi)u^{r_2}(\xi)\,d\xi\right)^{r_1} d\tau\,ds \\
&\leqslant c_1\varepsilon_1^{r_1} \int_0^1 \int_0^1 s(1-s)G(s,\tau)p_1(\tau)\,d\tau\,ds \\
&\quad\cdot \left(\int_0^1 \xi(1-\xi)p_2(\xi)\,d\xi\right)^{r_1} \cdot \|u\|^{r_1 r_2} \\
&\leqslant \|u\|^{r_1 r_2} \leqslant \|u\|, \quad \forall u \in \partial B_\delta \cap K,\ t \in [0,1].
\end{aligned}$$

Consequently,

$$\|Au\| \leqslant \|u\|, \quad \forall u \in \partial B_\delta \cap K. \tag{2.11}$$

On the other hand, by condition (H_3), we know that there exist $c_2 > 0$, $\varepsilon_2 > 0$, and $R_1 > 1$ such that

$$\begin{aligned}
&f(t,u) \geqslant \varepsilon_2 u^{l_1}, \quad \forall u \geqslant R_1,\ t \in \left[\frac{1}{4}, \frac{3}{4}\right], \\
&g(t,u) \geqslant c_2 u^{l_2}, \quad \forall u \geqslant R_1,\ t \in \left[\frac{1}{4}, \frac{3}{4}\right],
\end{aligned}$$

and satisfying

$$\frac{c_2}{4^{l_2+1}}\int_{1/4}^{3/4}\xi(1-\xi)\,d\xi \geqslant 1,$$

$$\left(\frac{c_2}{4^{l_2+1}}\right)^{l_1}\cdot\varepsilon_2\int_{1/4}^{3/4}\int_{1/4}^{3/4}G\left(\frac{1}{2},s\right)G(s,\tau)\,d\tau\,ds\cdot\left(\int_{1/4}^{3/4}\xi(1-\xi)\,d\xi\right)^{l_1}\geqslant 1.$$

Let $R>\max\{4R_1, R_1^{1/l_2}\}$. Then for $u\in\partial B_R\cap K$, we have

$$\min_{t\in[1/4,3/4]}u(t)\geqslant\frac{1}{4}\|u\|=\frac{1}{4}R>R_1$$

and

$$\int_0^1 G(\tau,\xi)g\big(\xi,u(\xi)\big)\,d\xi\geqslant\int_{1/4}^{3/4}G(\tau,\xi)g\big(\xi,u(\xi)\big)\,d\xi\geqslant c_2\int_{1/4}^{3/4}G(\tau,\xi)u^{l_2}(\xi)\,d\xi$$
$$\geqslant\frac{c_2}{4}\int_{1/4}^{3/4}G(\xi,\xi)u^{l_2}(\xi)\,d\xi\geqslant\frac{c_2}{4^{l_2+1}}\int_{1/4}^{3/4}\xi(1-\xi)\,d\xi\,\|u\|^{l_2}\geqslant\|u\|^{l_2}=R^{l_2}>R_1,$$
$$\forall\tau\in\left[\frac{1}{4},\frac{3}{4}\right].$$

Thus, we get

$$(Au)\left(\frac{1}{2}\right)=\int_0^1 G\left(\frac{1}{2},s\right)\int_0^1 G(s,\tau)f\left(\tau,\int_0^1 G(\tau,\xi)g\big(\xi,u(\xi)\big)\,d\xi\right)d\tau\,ds$$
$$\geqslant\int_{1/4}^{3/4}G\left(\frac{1}{2},s\right)\int_{1/4}^{3/4}G(s,\tau)f\left(\tau,\int_0^1 G(\tau,\xi)g\big(\xi,u(\xi)\big)\,d\xi\right)d\tau\,ds$$
$$\geqslant\varepsilon_2\int_{1/4}^{3/4}G\left(\frac{1}{2},s\right)\int_{1/4}^{3/4}G(s,\tau)\left(\int_0^1 G(\tau,\xi)g\big(\xi,u(\xi)\big)\,d\xi\right)^{l_1}d\tau\,ds$$
$$\geqslant\varepsilon_2\int_{1/4}^{3/4}G\left(\frac{1}{2},s\right)\int_{1/4}^{3/4}G(s,\tau)\left(\frac{c_2}{4^{l_2+1}}\int_{1/4}^{3/4}\xi(1-\xi)\,d\xi\cdot\|u\|^{l_2}\right)^{l_1}d\tau\,ds$$
$$=\left(\frac{c_2}{4^{l_2+1}}\right)^{l_1}\cdot\varepsilon_2\int_{1/4}^{3/4}\int_{1/4}^{3/4}G\left(\frac{1}{2},s\right)G(s,\tau)\,d\tau\,ds\cdot\left(\int_{1/4}^{3/4}\xi(1-\xi)\,d\xi\right)^{l_1}$$
$$\cdot\|u\|^{l_1l_2}$$
$$\geqslant\|u\|^{l_1l_2}\geqslant\|u\|,\quad\forall u\in\partial B_R\cap K.$$

Consequently,

$$\|Au\| \geqslant \|u\|, \quad \forall u \in \partial B_R \cap K. \tag{2.12}$$

By Lemmas 1, 2, (2.10)–(2.12), we obtain that A has a fixed point in $(\bar{B}_R \setminus B_\delta) \cap K$. Therefore, BVP (1.1) has a positive solution $(u, v) \in K \times K$ satisfying $u(t) > 0$, $v(t) > 0$ for $t \in (0, 1)$. □

Proof of Theorem 2. By condition (H_4), we know there exist $c_3 > 0$, $\varepsilon_3 > 0$, $N_1 > 0$, and $N_2 > 0$ such that

$$q_1(u) \leqslant c_3 u^{\alpha_1} + N_1, \quad q_2(u) \leqslant \varepsilon_3 u^{\alpha_2} + N_2, \quad u \in R^+,$$

and satisfying

$$(2\varepsilon_3)^{\alpha_1} c_3 \int_0^1 \int_0^1 s(1-s) G(s,\tau) p_1(\tau)\, d\tau\, ds \cdot \left(\int_0^1 \xi(1-\xi) p_2(\xi)\, d\xi \right)^{\alpha_1} < 1. \tag{2.13}$$

Therefore,

$$\begin{aligned}
(Au)(t) &\leqslant \int_0^1 G(t,s) \int_0^1 G(s,\tau) p_1(\tau) q_1\left(\int_0^1 G(\tau,\xi) g\big(\xi, u(\xi)\big)\, d\xi \right) d\tau\, ds \\
&\leqslant \int_0^1 G(t,s) \int_0^1 G(s,\tau) p_1(\tau) \left[c_3 \left(\int_0^1 G(\tau,\xi) g\big(\xi, u(\xi)\big)\, d\xi \right)^{\alpha_1} + N_1 \right] d\tau\, ds \\
&\leqslant N_1 \int_0^1 \int_0^1 G(s,s) G(s,\tau) p_1(\tau)\, d\tau\, ds \\
&\quad + c_3 \int_0^1 G(s,s) \int_0^1 G(s,\tau) p_1(\tau) \left(\int_0^1 G(\tau,\xi) p_2(\xi) q_2\big(u(\xi)\big)\, d\xi \right)^{\alpha_1} d\tau\, ds \\
&\leqslant N_1 \int_0^1 \int_0^1 s(1-s) G(s,\tau) p_1(\tau)\, d\tau\, ds \\
&\quad + c_3 \int_0^1 G(s,s) \int_0^1 G(s,\tau) p_1(\tau) \\
&\quad \cdot \left(\int_0^1 G(\tau,\xi) p_2(\xi) \big(\varepsilon_3 u^{\alpha_2}(\xi) + N_2\big)\, d\xi \right)^{\alpha_1} d\tau\, ds
\end{aligned}$$

$$
\begin{aligned}
&\leqslant N_1 \int_0^1 \int_0^1 s(1-s)G(s,\tau)p_1(\tau)\,d\tau\,ds \\
&\quad + c_3 \int_0^1 \int_0^1 s(1-s)G(s,\tau)p_1(\tau)\,d\tau\,ds \\
&\quad \cdot \left(\int_0^1 \xi(1-\xi)p_2(\xi)\,d\xi\right)^{\alpha_1} \bigl(\varepsilon_3\|u\|^{\alpha_2}+N_2\bigr)^{\alpha_1} \\
&\leqslant N_1 \int_0^1 \int_0^1 s(1-s)G(s,\tau)p_1(\tau)\,d\tau\,ds \\
&\quad + c_3 \int_0^1 \int_0^1 s(1-s)G(s,\tau)p_1(\tau)\,d\tau\,ds \\
&\quad \cdot \left(\int_0^1 \xi(1-\xi)p_2(\xi)\,d\xi\right)^{\alpha_1} \bigl[2^{\alpha_1}\bigl(\varepsilon_3^{\alpha_1}\|u\|^{\alpha_1\alpha_2}+N_2^{\alpha_1}\bigr)\bigr] \\
&\leqslant N_1 \int_0^1 \int_0^1 s(1-s)G(s,\tau)p_1(\tau)\,d\tau\,ds \\
&\quad + (2N_2)^{\alpha_1}c_3 \int_0^1 \int_0^1 s(1-s)G(s,\tau)p_1(\tau)\,d\tau\,ds \cdot \left(\int_0^1 \xi(1-\xi)p_2(\xi)\,d\xi\right)^{\alpha_1} \\
&\quad + (2\varepsilon_3)^{\alpha_1}c_3 \int_0^1 \int_0^1 s(1-s)G(s,\tau)p_1(\tau)\,d\tau\,ds \\
&\quad \cdot \left(\int_0^1 \xi(1-\xi)p_2(\xi)\,d\xi\right)^{\alpha_1} \|u\|^{\alpha_1\alpha_2}.
\end{aligned}
$$

By (2.13), we can choose $R_2 > 0$, which is sufficiently large such that

$$\|Au\| \leqslant \|u\|, \quad \forall u \in \partial B_{R_2} \cap K. \tag{2.14}$$

On the other hand, by condition (H_5), we know there exist $\varepsilon_4 > 0$, $c_4 > 0$, and $\rho \in (0,1)$ such that

$$
\begin{aligned}
&f(t,u) \geqslant \varepsilon_4 u^{\beta_1}, \quad (t,u) \in \left[\frac{1}{4},\frac{3}{4}\right] \times [0,\rho], \\
&g(t,u) \geqslant c_4 u^{\beta_2}, \quad (t,u) \in \left[\frac{1}{4},\frac{3}{4}\right] \times [0,\rho],
\end{aligned}
$$

and satisfying

$$\left(\frac{c_4}{4^{\beta_2+1}}\right)^{\beta_1}\varepsilon_4\int_{1/4}^{3/4}\int_{1/4}^{3/4}G\left(\frac{1}{2},s\right)G(s,\tau)\,d\tau\,ds\left(\int_{1/4}^{3/4}\xi(1-\xi)\,d\xi\right)^{\beta_1}\geqslant 1.$$

Since $q_2(0)=0$ and the continuity of $q_2(u)$, there exists $\varepsilon\in(0,\rho)$, which is sufficiently small such that

$$q_2(u)\leqslant\left(\int_0^1 G(\xi,\xi)p_2(\xi)\,d\xi\right)^{-1}\rho,\quad \forall u\in[0,\varepsilon].$$

Thus for $\forall u\in\partial B_\varepsilon\cap K$ and $\tau\in[0,1]$, we have

$$\begin{aligned}\int_0^1 G(\tau,\xi)g\big(\xi,u(\xi)\big)\,d\xi &\leqslant \int_0^1 G(\tau,\xi)p_2(\xi)q_2\big(u(\xi)\big)\,d\xi\\ &\leqslant\left(\int_0^1 G(\xi,\xi)p_2(\xi)\,d\xi\right)^{-1}\rho\int_0^1 G(\tau,\xi)p_2(\xi)\,d\xi\leqslant\rho.\end{aligned}$$

Therefore, we get

$$\begin{aligned}(Au)\left(\frac{1}{2}\right) &\geqslant \int_{1/4}^{3/4}G\left(\frac{1}{2},s\right)\int_{1/4}^{3/4}G(s,\tau)f\left(\tau,\int_0^1 G(\tau,\xi)g\big(\xi,u(\xi)\big)\,d\xi\right)d\tau\,ds\\ &\geqslant \varepsilon_4\int_{1/4}^{3/4}G\left(\frac{1}{2},s\right)\int_{1/4}^{3/4}G(s,\tau)\left(\int_0^1 G(\tau,\xi)g\big(\xi,u(\xi)\big)\,d\xi\right)^{\beta_1}d\tau\,ds\\ &\geqslant \varepsilon_4\int_{1/4}^{3/4}G\left(\frac{1}{2},s\right)\int_{1/4}^{3/4}G(s,\tau)\left(\int_{1/4}^{3/4}G(\tau,\xi)g\big(\xi,u(\xi)\big)\,d\xi\right)^{\beta_1}d\tau\,ds\\ &\geqslant \varepsilon_4\int_{1/4}^{3/4}G\left(\frac{1}{2},s\right)\int_{1/4}^{3/4}G(s,\tau)\left(\int_{1/4}^{3/4}G(\tau,\xi)c_4u^{\beta_2}(\xi)\,d\xi\right)^{\beta_1}d\tau\,ds\\ &\geqslant \left(\frac{c_4}{4^{\beta_2+1}}\right)^{\beta_1}\varepsilon_4\int_{1/4}^{3/4}\int_{1/4}^{3/4}G\left(\frac{1}{2},s\right)G(s,\tau)\,d\tau\,ds\\ &\quad\cdot\left(\int_{1/4}^{3/4}\xi(1-\xi)\,d\xi\right)^{\beta_1}\|u\|^{\beta_1\beta_2}\\ &\geqslant \|u\|^{\beta_1\beta_2}\geqslant\|u\|,\quad \forall u\in\partial B_\varepsilon\cap K.\end{aligned}$$

Consequently,

$$\|Au\| \geqslant \|u\|, \quad \forall u \in \partial B_\varepsilon \cap K. \tag{2.15}$$

By Lemmas 1, 2, (2.10), (2.14), and (2.15), we obtain that A has a fixed point in $(\bar{B}_{R_2} \setminus B_\varepsilon) \cap K$. Therefore, BVP (1.1) has a positive solution $(u, v) \in K \times K$ satisfying $u(t) > 0$, $v(t) > 0$ for $t \in (0, 1)$. $\square$

Proof of Theorem 3. By condition (H_6), for $\forall u \in \partial B_N \cap K$ and $\tau \in [0, 1]$, we get

$$\begin{aligned}
\int_0^1 G(\tau, \xi) g\bigl(\xi, u(\xi)\bigr)\, d\xi &\leqslant \int_0^1 G(\tau, \xi) p_2(\xi) q_2\bigl(u(\xi)\bigr)\, d\xi \\
&\leqslant M \int_0^1 G(\xi, \xi) p_2(\xi)\, d\xi = Mb,
\end{aligned}$$

where $M = \sup_{u \in [0,N]} q_2(u)$, b is the same as in (H_1).

Thus, we get

$$\begin{aligned}
(Au)(t) &\leqslant \int_0^1 G(t, s) \int_0^1 G(s, \tau) p_1(\tau) q_1\left(\int_0^1 G(\tau, \xi) g\bigl(\xi, u(\xi)\bigr)\, d\xi \right) d\tau\, ds \\
&\leqslant \sup_{u \in [0, Mb]} q_1(u) \int_0^1 \int_0^1 s(1-s) G(s, \tau) p_1(\tau)\, d\tau\, ds \\
&= a \cdot \sup_{u \in [0, Mb]} q_1(u) \leqslant N, \quad \forall u \in \partial B_N \cap K,\ t \in [0, 1],
\end{aligned}$$

where a is the same as in (H_1). Consequently,

$$\|Au\| \leqslant \|u\|, \quad \forall u \in \partial B_N \cap K. \tag{2.16}$$

On the other hand, by (H_3) and (H_5), for sufficiently large $R > N$ and sufficiently small $\varepsilon \in (0, N)$, (2.12) and (2.15) hold. In addition, by (2.10) and (2.16), we obtain A has at least one fixed point in $(\bar{B}_R \setminus B_N) \cap K$ and $(\bar{B}_N \setminus B_\varepsilon) \cap K$, respectively. Therefore, BVP (1.1) has at least two positive solutions $(u_1, v_1) \in K \times K$, $(u_2, v_2) \in K \times K$ satisfying $u_i(t) > 0$, $v_i(t) > 0$ for $t \in (0, 1)$ $(i = 1, 2)$. $\square$

3. Some examples

In the following we give some examples to illustrate the theorems obtained in Section 2.

Example 1. In BVP (1.1), let $f(t, v) = \frac{v^2}{\sqrt{1-t}}$, $g(t, u) = \frac{u^4}{\sqrt{t}}$, we can choose $p_1(t) = \frac{1}{\sqrt{1-t}}$, $q_1(v) = v^2$, $p_2(t) = \frac{1}{\sqrt{t}}$, $q_2(u) = u^4$. Then (H_1) is satisfied, where

$$a = \frac{1}{12}\int_0^1 (\tau - 2\tau^3 + \tau^4)\frac{1}{\sqrt{1-\tau}}\,d\tau \leqslant \frac{1}{12}\int_0^1 \frac{d\tau}{\sqrt{1-\tau}} = \frac{1}{12}\cdot 2 = \frac{1}{6},$$

$$b = \int_0^1 s(1-s)\frac{1}{\sqrt{s}}\,ds = \int_0^1 (s^{1/2} - s^{3/2}) = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}.$$

Let $r_1 = \frac{1}{2}$, $r_2 = 3$, $l_1 = \frac{1}{2}$, $l_2 = 3$. Then (H_2) and (H_3) are satisfied. Therefore, by Theorem 1, we know BVP (1.1) has at least one solution.

Example 2. In BVP (1.1), let $f(t,v) = \frac{v^{1/2}}{\sqrt{1-t}}$, $g(t,u) = \frac{u^{1/2}}{\sqrt{t}}$, we can choose $p_1(t) = \frac{1}{\sqrt{1-t}}$, $q_1(v) = v^{1/2}$, $p_2(t) = \frac{1}{\sqrt{t}}$, $q_2(u) = u^{1/2}$. Then (H_1) is satisfied. Let $\alpha_1 = \frac{1}{2}$, $\alpha_2 = 1$, $\beta_1 = \frac{1}{2}$, $\beta_2 = 1$. Then (H_4) and (H_5) are satisfied. Therefore, by Theorem 2, we know BVP (1.1) has at least one solution.

Example 3. In BVP (1.1), let $f(t,v) = \frac{1}{10\sqrt{1-t}}(v^2 + v^{1/2})$, $g(t,u) = \frac{1}{\sqrt{t}}(u^3 + u^{1/2})$, we can choose $p_1(t) = \frac{1}{\sqrt{1-t}}$, $q_1(v) = \frac{1}{10}(v^2 + v^{1/2})$, $p_2(t) = \frac{1}{\sqrt{t}}$, $q_2(u) = u^3 + u^{1/2}$, then (H_1) is satisfied, where $a \leqslant \frac{1}{6}$, $b = \frac{4}{15}$. Let $l_1 = \frac{1}{2}$, $l_2 = 2$, $\beta_1 = \frac{1}{2}$, $\beta_2 = 1$, $N = 1$, then (H_3) and (H_5) are satisfied, and

$$M = \sup_{u\in[0,1]} q_2(u) = \sup_{u\in[0,1]} (u^3 + u^{1/2}) = 2,$$

$$a \sup_{u\in[0,Mb]} q_1(u) \leqslant \sup_{u\in[0,8/15]} \frac{1}{10}(u^2 + u^{1/2}) \leqslant 1.$$

Thus (H_6) is satisfied. Therefore, by Theorem 3, we know BVP (1.1) has at least two solutions.

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References

[1] R.P. Agarwal, D. O'Regan, Nonlinear superlinear singular and nonsingular second order boundary value problems, J. Differential Equations 143 (1998) 60–95.
[2] R.P. Agarwal, D. O'Regan, Singular boundary value problems, Nonlinear Anal. 27 (1996) 645–656.
[3] D. O'Regan, Solvability of some fourth (and higher) order singular boundary value problems, J. Math. Anal. Appl. 161 (1991) 78–116.
[4] R. Ma, On the existence of positive solutions of fourth order ordinary differential equations, Appl. Anal. 59 (1995) 225–231.
[5] R. Dalmasso, Uniqueness theorem for some fourth order elliptic equations, Proc. Amer. Math. Soc. 123 (1995) 1177–1183.

[6] Y. Liu, Structure of a class of singular boundary value problem with superlinear effect, J. Math. Anal. Appl. 284 (2003) 64–75.
[7] R.P. Agarwal, D. O'Regan, Multiple solutions for a coupled system of boundary value problems, Dynam. Contin. Discrete Impuls. Systems 7 (2000) 97–106.
[8] R.P. Agarwal, D. O'Regan, A coupled system of boundary value problems, Appl. Anal. 69 (1998) 381–385.
[9] R.P. Agarwal, D. O'Regan, P.J.Y. Wong, Constant-sign solutions of a system of Fredholm integral equations, Acta Appl. Math. 80 (2004) 57–94.
[10] S. An, Y. An, Existence of positive solution of a class ordinary differential system, J. Engrg. Math. 21 (2004) 70–74 (in Chinese).
[11] Y. An, Nonlinear perturbations of a coupled system of steady state suspension bridge equations, Nonlinear Anal. 51 (2002) 1285–1292.
[12] Y. Liu, B. Yan, Multiple solutions of singular boundary value problems for differential systems, J. Math. Anal. Appl. 287 (2003) 540–556.
[13] A.C. Lazer, P.J. Mckenna, Large-amplitude periodic oscillations in suspension bridge: some new connections with nonlinear analysis, SIAM Rev. 32 (1990) 537–578.
[14] D. Guo, V. Lakshmikantham, X. Liu, Nonlinear Integral Equations in Abstract Spaces, Kluwer Academic, Dordrecht, 1996.